Theorem: Let $f: \{\pm 1\}^n \to \{\pm 1\}$. Then,

$$\forall k: \quad \sum_{|S|=k} |\hat{f}(S)| \le 2^k \cdot \mathbb{E}_{S \sim \mathcal{D}_f}\left[\binom{|S|}{k}\right]$$

Proof:

Recall:

$$D_i f(x) = \sum_{S: S \ni i} \hat{f}(S)\, \chi_{S \setminus \{i\}}(x) = \frac{f(x|_{x_i=1}) - f(x|_{x_i=-1})}{2}$$

$$I_i(f) \triangleq \mathbb{E}_{x \in_R \{\pm 1\}^n}\left[(D_i f(x))^2\right] = \sum_{S \ni i} \hat{f}(S)^2$$

On the other hand, $\mathbb{E}_{x \in_R \{\pm 1\}^n}[D_i f(x)] = \hat{f}(\{i\})$.

Let's generalize to higher moments:

$$T = \{i_1, i_2, \dots, i_k\}$$

$$D_T f(x) \triangleq D_{i_k} \cdots D_{i_2} D_{i_1} f(x)$$

$$= \sum_{S: T \subseteq S} \hat{f}(S)\, \chi_{S \setminus T}(x)$$

$$= \frac{1}{2^{|T|}} \cdot \sum_{z \in \{\pm 1\}^k} f(x|_{x_T \leftarrow z}) \cdot z_1 z_2 \cdots z_k$$

Then, $\mathbb{E}[D_T f(x)^2] = \sum_{S: T \subseteq S} \hat{f}(S)^2$

and $\mathbb{E}[D_T f(x)] = \hat{f}(T)$.

$$L_{1,k}(f) = \sum_{|T|=k} |\hat{f}(T)| = \sum_{|T|=k} \left| \mathbb{E}_x [D_T f(x)] \right|$$

$$\le \sum_{|T|=k} \mathbb{E}_x \left[|D_T f(x)| \right]$$

Since f is Boolean $D_T f(x) = \frac{\text{integer}}{2^k}$ $\longrightarrow$

$$\le \sum_{|T|=k} \mathbb{E}_x \left[D_T f(x)^2 \cdot 2^k \right]$$

$$= 2^k \cdot \sum_{|T|=k} \sum_{S : T \subseteq S} \hat{f}(S)^2$$

$$= 2^k \cdot \sum_{S \subseteq [n]} \hat{f}(S)^2 \cdot \binom{|S|}{k}$$

$\blacksquare$

$$\boxed{\textbf{Theorem: } \text{If } f : \{\pm 1\}^n \to \{\pm 1\} \text{ and } \deg(f) = d \text{ then } \forall S: \ \hat{f}(S) = \frac{\text{integer}}{2^d}.}$$

Proof:

Case $|S| = d$.

$$D_S f(x) = \hat{f}(S) = \frac{\text{integer}}{2^d}.$$

Case $|S| = d-1$:

$$\underbrace{D_S f(x)}_{\frac{\text{integer}}{2^{d-1}}} = \hat{f}(S) + \underbrace{\sum_{T \supseteq S} \hat{f}(T) \chi_{T \setminus S}(x)}_{\frac{\text{integer}}{2^d}}.$$

Case $|S| < d-1$: similarly ... (i.e., by induction)

$\blacksquare$